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On the Lebesgue Constants for Cardinal $\mathscr{L}$-Spline Interpolation

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1. INTRODUCTION AND SUMMARY

Throughout this paper p_{2k+1} denotes the monic polynomial $p_{2k+1}(x) = x(x^2-\alpha_1)\cdots(x^2-\alpha_k)$, where $\alpha_1,..., \alpha_k$ are real numbers such that $0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_k$. The linear differential operator having p_{2k+1} as its characteristic polynimial is denoted by $\mathscr{L}_{2k+1}$, i.e., $\mathscr{L}_{2k+1}(D) = p_{2k+1}(D)$, where D is the ordinary first-order differentiation operator.

A complex-valued function s is called a *cardinal $\mathscr{L}$-spine* with respect to $\mathscr{L}_{2k+1}$ if it satisfies the conditions

$$\begin{aligned} &\text{(i)} \quad s \in C^{(2k-1)}(\mathbb{R}), \\ &\text{(ii)} \quad \mathscr{L}_{2k+1}s(t) = 0 \qquad (\nu < t < \nu+1,\ \nu = 0, \pm 1, \pm 2,...). \end{aligned} \tag{1.1}$$

The set of cardinal $\mathscr{L}$-splines with respect to $\mathscr{L}_{2k+1}$ is denoted by S_{2k+1}. Obviously, S_{2k+1} depends on $\alpha_1,..., \alpha_k$; this, however, is suppressed in our notation. The following interpolation property holds.

LEMMA 1.1 (Michelli [4]). *Let $(y_\nu)_{-\infty}^{\infty}$ be a bounded sequence of complex numbers. Then a unique bounded function $s \in S_{2k+1}$ exists such that*

$$s(\nu + \tfrac{1}{2}) = y_\nu \qquad (\nu = 0, \pm 1, \pm 2,...). \tag{1.2}$$

The boundedness of the interpolant s in Lemma 1.1 is required to ensure the unicity of s.

Let $\mathscr{S}_{2k+1}$ be the linear operator mapping the set of bounded sequences $\mathbf{y} = (y_\nu)_{-\infty}^{\infty}$ onto the set of bounded functions in S_{2k+1} by way of inter-

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polation according to (1.2). The purpose of this paper is to study the asymptotic behaviour of the operator norm

$$\|\mathscr{S}_{2k+1}\| = \sup_{\mathbf{y} \neq \mathbf{0}} \frac{\|\mathscr{S}_{2k+1}\mathbf{y}\|_\infty}{\|\mathbf{y}\|_\infty} \tag{1.3}$$

as $k \to \infty$.

Taking in particular the sequence $(y_\nu) = (\delta_{\nu,0})$ in (1.2) we obtain the so-called fundamental solution L_{2k+1} of the interpolation problem. In Schoenberg [8] it is shown that $|L_{2k+1}(t)| < Ae^{-\alpha|t|}$ $(t \in \mathbb{R})$ for appropriate positive constants A and α. Hence, for any bounded sequence $\mathbf{y} = (y_\nu)_{-\infty}^{\infty}$, the corresponding bounded interpolant $\mathscr{S}_{2k+1}\mathbf{y}$ may be written in the form

$$\mathscr{S}_{2k+1}\mathbf{y}(t) = \sum_{\nu=-\infty}^{\infty} y_\nu L_{2k+1}(t-\nu) \qquad (-\infty < t < \infty). \tag{1.4}$$

It immediately follows from 1.4 that

$$\|\mathscr{S}_{2k+1}\| \leqslant \sup_{t \in \mathbb{R}} \bar{L}_{2k+1}(t),$$

where

$$\bar{L}_{2k+1}(t) = \sum_{\nu=-\infty}^{\infty} |L_{2k+1}(t-\nu)| \tag{1.5}$$

is the Lebesgue function associated with the given cardinal interpolation problem.

In Section 3 it is proved that on $[-\frac{1}{2}, \frac{1}{2}]$ the function $\bar{L}_{2k+1}$ coincides with the cardinal $\mathscr{L}$-spline

$$\tilde{L}_{2k+1}(t) = \sum_{\nu=-\infty}^{\infty} \tilde{y}_\nu L_{2k+1}(t-\nu) \qquad (-\infty < t < \infty), \tag{1.6}$$

where

$$\begin{aligned} \tilde{y}_\nu &= (-1)^\nu \qquad (\nu = 0, 1, 2, \ldots), \\ &= (-1)^{\nu+1} \qquad (\nu = -1, -2, \ldots). \end{aligned} \tag{1.7}$$

We also show that

$$\|\mathscr{S}_{2k+1}\| = \tilde{L}_{2k+1}(0). \tag{1.8}$$

In view of this operator norm $\|\mathscr{S}_{2k+1}\|$ (cf. (1.3)) is also called the Lebesgue constant for the interpolation problem. Our study of the asymptotic behaviour of $\|\mathscr{S}_{2k+1}\|$ $(k \to \infty)$ is based on an integral representation of $\|\mathscr{S}_{2k+1}\|$; cf. also Section 3. In order to derive this representation, some known results in the theory of cardinal $\mathscr{L}$-splines are needed; these are collected in Section 2. Finally, the asymptotic behaviour of $\|\mathscr{S}_{2k+1}\|$ is studied in Section 4. The following result is obtained.

Let

$$\beta_k = \frac{2}{\pi} + 4\pi \sum_{j=1}^{k} \frac{1}{\alpha_j + \pi^2}|,$$

and let γ denote Euler's constant. It is shown that

$$\|\mathcal{S}_{2k+1}\| = \frac{2}{\pi}(\ln \beta_k + 3 \ln 2 - \ln \pi + \gamma) + \mathcal{O}(\beta_k^{-2}) \qquad (k \to \infty),$$

as $\beta_k \to \infty$ $(k \to \infty)$. If the sequence (β_k) converges then it is proved that $\|\mathcal{S}_{2k+1}\|$ converges as well.

2. Preliminaries

Let the polynomial $\tilde{p}_{2k+1}$ be defined by

$$\tilde{p}_{2k+1}(z) = (z-1)(z-e^{-\sqrt{\alpha_1}})(z-e^{\sqrt{\alpha_1}}) \cdots (z-e^{-\sqrt{\alpha_k}})(z-e^{\sqrt{\alpha_k}}), \quad (2.1)$$

where $z \in \mathbb{C}$.

For all $z \in \mathbb{C}$ with $\tilde{p}_{2k+1}(z) \neq 0$ and for all $t \in \mathbb{R}$ the function $\psi(z, t)$ is then defined by

$$\psi(z, t) = \frac{\tilde{p}_{k+1}(z)}{2\pi i} \oint_C \frac{e^{t\zeta}}{(z - e^{\zeta})\, p_{2k+1}(\zeta)}\, d\zeta, \quad (2.2)$$

where p_{2k+1} is given in Section 1, and where C is any contour in the complex plane surrounding the zeros of p_{2k+1} but excluding the zeros of $\zeta \mapsto z - e^{\zeta}$.

In the sequel the following properties of $\psi(z, t)$ are needed; they are contained in ter Morsche [6] as well as in Michelli [4], where, apart from a normalisation factor, $\psi(z, t)$ is also used.

One has

$$t \mapsto \psi(z, t) \in \mathrm{Ker}(\mathcal{L}_{2k+1}), \qquad \text{the kernel of } \mathcal{L}_{2k+1}, \quad (2.3)$$

$$\left(\frac{\partial}{\partial t}\right)^j \psi(z, t)\bigg|_{t=1} = z \left(\frac{\partial}{\partial t}\right)^j \psi(z, t)\bigg|_{t=0} \qquad (j = 0, 1, ..., 2k-1), \quad (2.4)$$

$$\psi(z, 1-t) = z^{2k} \psi(z^{-1}, t), \quad (2.5)$$

$$\psi(z, t) = \sum_{j=0}^{2k} A_j(t)\, z^j, \qquad \text{with} \quad A_j \in \mathrm{Ker}(\mathcal{L}_{2k+1}),\ A_{2k}(t) > 0 \quad (t \neq 0), \qquad A_{2k}(0) = 0. \quad (2.6)$$

Apart from these relations the following property of $\psi(z, t)$ is of interest.

LEMMA 1.2 (Michelli [4]). *If* $z<0$ *the function* $t \mapsto \psi(z, t)$ *has precisely one zero in* $(0, 1]$. *Furthermore, if* $t \in [0, 1)$ *then the polynomial* $z \mapsto \psi(z, t)$ *has only real zeros; these zeros are negative and simple.*

The polynomial $z \mapsto \psi(z, t)$ is usually called the exponential $\mathscr{L}$-polynomial, and in case $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ it is the well-known Euler–Fröbenius polynomial of degree at most $2k$ (cf. ter Morsche [6, p. 62]). From (2.4) it follows that $\psi(z, 1) = z\psi(z, 0)$. Therefore, by Lemma 2.1, $\psi(z, 1)$ has $2k-1$ negative simple zeros and, in addition, $z=0$ is also a zero.

Let the zeros of $z \mapsto \psi(z, t)$ $(t \in (0, 1])$ be denoted by $\lambda_1(t),..., \lambda_{2k}(t)$ with

$$-\infty < \lambda_1(t) < \lambda_2(t) < \cdots < \lambda_{2k}(t) \leqslant 0.$$

In Schoenberg [8] it is shown that the functions $t \mapsto \lambda_i(t)$ $(i = 1,..., 2k)$ are increasing on $(0, 1]$, satisfying the inequalities

$$\begin{gathered}\lambda_{i-1}(1) < \lambda_i(t_1) < \lambda_i(t_2) < \lambda_i(1) \leqslant 0,\\ \text{where } 0 < t_1 < t_2 < 1 \text{ and, by definition, } \lambda_0(1) = -\infty.\end{gathered} \tag{2.7}$$

In the polynomial case, i.e., the case $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$, the inequalities (2.7) are already contained in ter Morsche [5].

In view of (2.5) the zeros of $\psi(z, \frac{1}{2})$ are ordered as

$$\begin{gathered}\lambda_1(\tfrac{1}{2}) < \cdots < \lambda_k(\tfrac{1}{2}) < -1 < \lambda_{k+1}(\tfrac{1}{2}) < \cdots < \lambda_{2k}(\tfrac{1}{2}) < 0,\\ \lambda_{k+i}(\tfrac{1}{2})\,\lambda_{k-i+1}(\tfrac{1}{2}) = 1 \qquad (i = 0, 1,..., k).\end{gathered} \tag{2.8}$$

According to ter Morsche [6, p. 68] the relation

$$\begin{gathered}\sum_{j=0}^{2k} A_j\left(\frac{1}{2}\right) s(\mu + j + t) = \sum_{j=0}^{2k} A_j(t)\, y_{\mu+j}\\ (0 \leqslant t < 1, \mu = 0, \pm 1,...)\end{gathered} \tag{2.9}$$

holds for all functions $s \in S_{2k+1}$ satisfying (1.2); here the functions A_j are given by (2.6).

Relation (2.9) may be considered as a linear difference equation for the unknown sequence $(s(\mu+t))_{\mu=-\infty}^{\infty}$ having $\psi(z, \frac{1}{2})$ as its characteristic polynomial.

We know, however, that the $\psi(z, \frac{1}{2})$ is a polynomial of degree $2k$ with $2k$ distinct negative zeros. Since, in view of (2.8), $\psi(-1, \frac{1}{2}) \neq 0$, the polynomial $\psi(z, \frac{1}{2})$ has no zeros on the unit circle in the complex plane, and therefore Lemma 3.4.1 of ter Morsche [6, p. 74] may be applied to (2.9). This yields the following result.

LEMMA 2.1. *Let $(y_\nu)_{-\infty}^{\infty}$ be a bounded sequence of complex numbers. Then a unique bounded function $s \in S_{2k+1}$ exists satisfying (1.2). Moreover, this interpolating function s can be written in the form*

$$s(\mu+t)=\sum_{j=-\infty}^{\infty} \omega_j(t)\, y_{\mu+j} \qquad (0 \leqslant t<1, \mu=0, \pm 1,...) \tag{2.10}$$

where $\omega_j(t)$ is given by the contour integral

$$\omega_j(t)=\frac{1}{2\pi i} \oint_{|z|=1} \frac{\psi(z, t)}{z^{j+1}\psi(z, \frac{1}{2})}\, dz \qquad (j=0, \pm 1, \pm 2,...). \tag{2.11}$$

3. THE LEBESGUE FUNCTION AND AN INTEGRAL REPRESENTATION OF $\|\mathscr{S}_{2k+1}\|$

An application of formula (2.10) to the particular sequence $(y_\nu)=(\delta_{\nu,0})$ yields the fundamental solution L_{2k+1} as introduced in Section 1. In view of Lemma 2.1 one has

$$L_{2k+1}(t-\mu)=\frac{1}{2\pi i} \oint_{|z|=1} \frac{\psi(z, t)}{z^{\mu+1}\psi(z, \frac{1}{2})}\, dz \qquad (0 \leqslant t \leqslant 1, \mu=0, \pm 1, \pm 2,...). \tag{3.1}$$

Using the residue theorem and (2.8), we obtain the representation

$$L_{2k+1}(t-\mu)=\sum_{l=k+1}^{2k} \frac{\psi(\lambda_l(\frac{1}{2}), t)}{(\lambda_l(\frac{1}{2}))^{\mu+1}\psi_z(\lambda_l(\frac{1}{2}), \frac{1}{2})} \qquad (0 \leqslant t<1, \mu=-1, -2,...), \tag{3.2}$$

here ψ_z denotes the partial derivative of $\psi(z, t)$ with respect to z. It follows from (2.7) that

$$\begin{aligned} \operatorname{sgn}\left(\frac{\psi(\lambda_l(\frac{1}{2}), t)}{\psi_z(\lambda_l(\frac{1}{2}), \frac{1}{2})}\right) &= -1 \qquad (\tfrac{1}{2}<t \leqslant 1),\\ &= 0 \qquad (t=\tfrac{1}{2}),\\ &= 1 \qquad (0 \leqslant t<\tfrac{1}{2}). \end{aligned} \tag{3.3}$$

Consequently,

$$\operatorname{sgn} L_{2k+1}(t-\mu)=(-1)^{\mu} \operatorname{sgn}(t-\tfrac{1}{2}) \qquad (0 \leqslant t<1, \mu=-1, -2,...). \tag{3.4}$$

Since, by Lemma 2.1, the function L_{2k+1} is uniquely determined, one has

$$L_{2k+1}(\tfrac{1}{2}+t)=L_{2k+1}(\tfrac{1}{2}-t) \qquad (-\infty<t<\infty). \tag{3.5}$$

Therefore

$$\operatorname{sgn} L_{2k+1}(t-\mu) = (-1)^{\mu} \operatorname{sgn}(\tfrac{1}{2}-t) \qquad (0<t\leqslant 1, \mu=1, 2,...). \quad (3.6)$$

Taking $\mu=0$ and applying the residue theorem, we obtain

$$L_{2k+1}(t) = \frac{\psi(0, t)}{\psi(0, \frac{1}{2})} + \sum_{l=k+1}^{2k} \frac{\psi(\lambda_l(\frac{1}{2}), t)}{\lambda_l(\frac{1}{2})\, \psi_z(\lambda_l(\frac{1}{2}), \frac{1}{2})} \qquad (0\leqslant t<1). \quad (3.7)$$

From (2.6) it follows that $\psi(0, t)\,\psi^{-1}(0, \frac{1}{2})>0$ $(t\in[0, 1))$. Using this and formulae (2.8), (3.3) we conclude that $L_{2k+1}(t)>0$ $(t\in[\frac{1}{2}, 1))$. Hence, in view of (3.5),

$$\operatorname{sgn}(L_{2k+1}(t)) = 1 \qquad (0<t<1). \quad (3.8)$$

The fundamental solution L_{2k+1} thus changes sign at the points $\nu+\frac{1}{2}$ $(\nu=\pm 1, \pm 2,...)$, and these points are the only zeros of L_{2k+1}.

Therefore, on the interval $[-\frac{1}{2}, \frac{1}{2}]$ the Lebesgue function $\bar{L}_{2k+1}$ as given by (1.5) coincides with the function $\tilde{L}_{2k+1}$ defined by (1.6). Having established this, our next goal is to show that $\|\mathscr{L}_{2k+1}\| = \tilde{L}_{2k+1}(0)$ holds. To this end we introduce the function $L_{2k+1}^{[n]}$ $(n\in\mathbb{N})$, being the unique bounded cardinal $\mathscr{L}$-spline in S_{2k+1} interpolating the periodic sequence

$$\begin{aligned} y_\nu^{[n]} &= (-1)^\nu && (\nu=0, 1,..., 2n), \\ y_{\nu+2n+1}^{[n]} &= \tilde{y}_\nu^{[n]} && (\nu=0, \pm 1, \pm 2,...). \end{aligned} \quad (3.9)$$

We emphasize that $y_\nu^{[n]} = y_{-\nu}^{[n]}$ $(\nu\in\mathbb{Z})$. Consequently, the unicity of $L_{2k+1}^{[n]}$ implies that $L_{2k+1}^{[n]}$ is an even and periodic function with period $2n+1$. Since (cf. (1.7))

$$y_\nu^{[n]} = \tilde{y}_\nu \qquad (\nu=-2n, -2n+1,..., 2n)$$

one has

$$\lim_{n\to\infty} L_{2k+1}^{[n]}(t) = \tilde{L}_{2k+1}(t),$$

uniformly on every compact interval of $\mathbb{R}$. Therefore (1.8) will be established if it is shown that

$$L_{2k+1}^{[n]}(0) = \max_{0\leqslant t\leqslant 1/2} L_{2k+1}^{[n]}(t). \quad (3.10)$$

This assertion may be proved as follows. Since $L_{2k+1}^{[n]}$ is an even function having at least $2n$ zeros in $(\frac{1}{2}, 2n+\frac{1}{2})$, its derivative $L_{2k+1}'^{[n]}$ has at least $2n-1$ zeros in $(\frac{1}{2}, 2n+\frac{1}{2})$, where, in addition,

$$L_{2k+1}'^{[n]}(0) = L_{2k+1}'^{[n]}(2n+1) = 0.$$

In order to prove that these zeros are the only zeros of $L'^{[n]}_{2k+1}$ on $[0, 2n+1]$, we use a generalized version of Rolle's theorem (cf. ter Morsche [6, Lemma 1.4.11]). Also taking into account that the functions involved, together with their $(2k-1)$st derivatives, are periodic with period $2n+1$, and the fact that

$$(D-\sqrt{\alpha_k}\,I)(D^2-\alpha_{k+1}I)\cdots(D^2-\alpha_1 I)\,L'^{[n]}_{2k+1}$$

has at most $2n$ sign changes in $(0, 2n+1)$, implies that $L'^{[n]}_{2k+1}$ has at most $2n-1$ zeros in $(0, 2n+1)$, it follows that $L'^{[n]}_{2k+1}$ has precisely $2n-1$ zeros in $(0, 2n+1)$, all of which are contained in the subinterval $(\frac{1}{2}, 2n+\frac{1}{2})$.

In view of $L^{[n]}_{2k+1}(\nu+\frac{1}{2})=(-1)^\nu$ $(\nu=0, 1, 2, \ldots, 2n)$ we obtain that $L'^{[n]}_{2k+1}(t)\leqslant 0$ in $(0, \frac{1}{2}]$. Hence (3.10) holds, which implies that $\|\mathscr{S}_{2k+1}\| = \tilde{L}_{2k+1}(0)$.

An integral representation of $\|\mathscr{S}_{2k+1}\|$ is now obtained as follows. We recall (cf. (1.6), (1.7)) that $\tilde{L}_{2k+1}$ is the unique bounded cardinal $\mathscr{L}$-spline interpolating the sequence $(\tilde{y}_\nu)$. Formula (2.10) combined with (2.11) yields

$$\tilde{L}_{2k+1}(0)=\frac{1}{2\pi i}\left(\sum_{j=-\infty}^{-1}\oint_{|z|=1-\varepsilon}\frac{(-1)^{j+1}\psi(z,0)}{z^{j+1}\psi(z,\frac{1}{2})}\,dz + \sum_{j=0}^{\infty}\oint_{|z|=1+\varepsilon}\frac{(-1)^{j}\psi(z,0)}{z^{j+1}\psi(z,\frac{1}{2})}\,dz\right),$$

where ε is chosen so small that $\psi(z, \frac{1}{2})$ has no zeros in the ring $1-2\varepsilon < |z| < 1+2\varepsilon$. Consequently,

$$\tilde{L}_{2k+1}(0)=\frac{1}{2\pi i}\left(\oint_{|z|=1-\varepsilon}\frac{\psi(z,0)}{(1+z)\,\psi(z,\frac{1}{2})}\,dz + \oint_{|z|=1+\varepsilon}\frac{\psi(z,0)}{(1+z)\,\psi(z,\frac{1}{2})}\,dz\right).$$

It easily follows from (2.4) and (2.5) that $\psi(-1, 0)=0$. Hence, by (1.8), we obtain an integral representation of the form

$$\|\mathscr{S}_{2k+1}\| = \frac{1}{\pi i}\oint_{|z|=1}\frac{\psi(z,0)}{(1+z)\,\psi(z,\frac{1}{2})}\,dz. \tag{3.11}$$

This formula will now be used to study the asymptotic behaviour of $\|\mathscr{S}_{2k+1}\|$. With respect to the polynomial case, the contour integral representation (3.11) was derived by G. Meinardus and G. Merz [3], who studied the norm of some periodic spline interpolation operators.

4. The Asymptotic Behaviour of $\|\mathscr{S}_{2k+1}\|$

We first observe that the sum of the residues of the function

$$\zeta \mapsto \frac{e^{t\zeta}}{(z-e^{\zeta})\, p_{2k+1}(\zeta)}$$

is zero in case $0 \leqslant t \leqslant 1$ as can be shown rather easily. Consequently, if $\varphi \neq 0 \pmod{2\pi}$, (2.2) yields

$$\psi(e^{i\varphi}, t) = \tilde{p}_{2k+1}(e^{i\varphi}) \sum_{m=-\infty}^{\infty} \frac{e^{i(t-1)(2m\pi+\varphi)}}{p_{2k+1}(2mi\pi + i\varphi)}.$$

Recalling that $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k$ (cf. (2.1)), we define the polynomial q_{2k+1} by

$$q_{2k+1}(z) = z(z^2+\alpha_1)\cdots(z^2+\alpha_k).$$

Since

$$p_{2k+1}(iz) = (-1)^k i q_{2k+1}(z)$$

one has

$$\frac{\psi(e^{i\varphi}, 0)}{\psi(e^{i\varphi}, \frac{1}{2})} = e^{-i\varphi/2} \frac{\sum_{m=-\infty}^{\infty} q_{2k+1}^{-1}(\varphi + 2m\pi)}{\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}^{-1}(\varphi + 2m\pi)}.$$

Substituting $z = e^{i(\pi-\tau)}$ in (3.11), we then obtain

$$\|\mathscr{S}_{2k+1}\| = \frac{1}{\pi}\int_0^{\pi} \frac{\sum_{m=-\infty}^{\infty} q_{2k+1}^{-1}((2m+1)\pi - \tau)}{\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}^{-1}((2m+1)\pi-\tau)} \frac{d\tau}{\sin(\tau/2)}. \quad (4.1)$$

Now let $u_{m,k}^{\pm}$ $(m = 0, 1, ...)$ be define by

$$u_{m,k}^{\pm}(\tau) = q_{2k+1}^{-1}((2m+1)\pi - \tau) \pm q_{2k+1}^{-1}((2m+1)\pi+\tau) \qquad (0 \leqslant \tau \leqslant \pi).$$

One easily verifies that

$$\sum_{m=-\infty}^{\infty} q_{2k+1}^{-1}((2m+1)\pi - \tau) = \sum_{m=0}^{\infty} u_{m,k}(\tau),$$

$$\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}^{-1}((2m+1)\pi - \tau) = \sum_{m=0}^{\infty} (-1)^m u_{m,k}^{+}(\tau).$$

Define the functions R_k^+, R_k^{-1}, and v_k on $[0, \pi]$ by

$$R_k^+(\tau) = q_{2k+1}(\pi - \tau) \sum_{m=1}^{\infty} (-1)^m u_{m,k}^+(\tau),$$

$$R_k^-(\tau) = q_{2k+1}(\pi - \tau) \sum_{m=1}^{\infty} u_{m,k}^-(\tau), \tag{4.2}$$

$$v_k(\tau) = q_{2k+1}(\pi - \tau)\, q_{2k+1}^{-1}(\pi + \tau).$$

In view of (4.1) we then have

$$\|\mathscr{S}_{2k+1}\| = \frac{1}{\pi} \int_0^{\pi} \frac{1 - v_k(\tau) + R_k^-(\tau)}{1 + v_k(\tau) + R_k^+(\tau)} \frac{d\tau}{\sin(\tau/2)}. \tag{4.3}$$

Let the increasing sequence $(\omega_k)_1^\infty$ be defined by

$$\omega_k = \sum_{j=1}^{k} (\alpha_j + 1)^{-1}. \tag{4.4}$$

From now on we distinguish between two cases, i.e.,

$$\lim_{k \to \infty} \omega_k = \infty \qquad \text{and} \qquad \lim_{k \to \infty} \omega_k < \infty.$$

I. $\lim_{k \to \infty} \omega_k = \infty$.

We first give a couple of assertions concerning the behaviour of the functions $u_{m,k}^-$ and $u_{m,k}^+$ as $k \to \infty$. Their verification involves staightforward, but rather tedious, computations, which are omitted here. The two relations are: a positive constant c exists such that for all $m \in \mathbb{N}$ and all $\tau \in [0, \pi]$

$$\begin{aligned} q_{2k+1}(\pi - \tau)\, u_{m,k}^-(\tau) &= \tau m^{-2} \mathcal{O}(e^{-c\omega_k}) \\ q_{2k+1}(\pi - \tau)\, u_{m,k}^+(\tau) &= m^{-3} \mathcal{O}(e^{-c\omega_k}) \end{aligned} \qquad (k \to \infty) \tag{4.5}$$

uniformly in m and τ. From (4.2) and (4.5) it immediately follows that

$$\begin{aligned} R_k^-(\tau) &= \tau \mathcal{O}(e^{-c\omega_k}) \\ R_k^+(\tau) &= \mathcal{O}(e^{-c\omega_k}) \end{aligned} \qquad (k \to \infty) \tag{4.6}$$

uniformly in τ. Since in view of (4.2) one has $v_k(\tau) \geq 0$ on $[0, \pi]$, it follows from (4.3) and (4.6) that

$$\|\mathscr{S}_{2k+1}\| = \frac{1}{\pi} \int_0^{\pi} \frac{1 - v_k(\tau)}{1 + v_k(\tau)} \frac{d\tau}{\sin(\tau/2)} (1 + \mathcal{O}(e^{-c\omega_k})) + \mathcal{O}(e^{-c\omega_k}) \tag{4.7}$$

as $k \to \infty$. In order to analyze (4.7), it is convenient to write v_k in the form

$$v_k(\tau) = \exp\left[\ln\left(\frac{\pi - \tau}{\pi + \tau}\right) + \sum_{j=1}^{k} \ln\left(\frac{\alpha_j + (\pi - \tau)^2}{\alpha_j + (\pi + \tau)^2}\right)\right].$$

Hence,

$$\ln v_k(\tau) = \ln\left(\frac{1 - \tau/\pi}{1 + \tau/\pi}\right) + \sum_{j=1}^{k} \ln\left(\frac{1 - 2\pi\tau(\alpha_j + \pi^2 + \tau^2)^{-1}}{1 + 2\pi\tau(\alpha_j + \pi^2 + \tau^2)^{-1}}\right).$$

We observe that $0 < \tau < \pi$ implies

$$0 \leqslant 2\pi\tau(\alpha_j + \pi^2 + \tau^2)^{-1} \leqslant 2\pi\tau(\pi^2 + \tau^2)^{-1} < 1.$$

An application of the Taylor expansion

$$\ln\left(\frac{1 - t}{1 + t}\right) = -2 \sum_{l=0}^{\infty} \frac{t^{2l+1}}{2l+1} \qquad (-1 \leqslant t < 1)$$

now yields

$$v_k(\tau) = \exp(-\tau g_k(\tau) - \tau^3 h_k(\tau)) \qquad (0 \leqslant \tau < \pi), \tag{4.8}$$

where

$$\begin{aligned} g_k(\tau) &= \frac{2}{\pi} + \sum_{j=1}^{k} \frac{4\pi}{\alpha_j + \pi^2 + \tau^2}, \\ h_k(\tau) &= 2\left(\sum_{l=1}^{\infty} \pi^{-2l-1} \frac{\tau^{2l-2}}{2l+1} + \sum_{j=1}^{k} \sum_{l=1}^{\infty} \left(\frac{2\pi}{\alpha_j + \pi^2 + \tau^2}\right)^{2l+1} \frac{\tau^{2l-2}}{2l+1}\right). \end{aligned} \tag{4.9}$$

Apparently, the function g_k satisfies on $[0, \pi)$ the inequalities

$$g_k(\tau) > \sum_{j=1}^{k} \frac{4\pi}{\alpha_j + \pi^2 + \tau^2} \geqslant \sum_{j=1}^{k} \frac{4\pi}{\alpha_j + 2\pi^2} \geqslant \frac{2}{\pi}\,\omega_k. \tag{4.10}$$

Since

$$\begin{aligned} &\sum_{j=1}^{k} \sum_{l=1}^{\infty} \left(\frac{2\pi}{\alpha_j + \pi^2 + \tau^2}\right)^{2l+1} \frac{\tau^{2l-2}}{2l+1} \\ &\quad = \sum_{j=1}^{k} \left(\frac{2\pi}{\alpha_j + \pi^2 + \tau^2}\right)^3 \sum_{l=1}^{\infty} \frac{1}{2l+1} \left(\frac{2\pi\tau}{\alpha_j + \pi^2 + \tau^2}\right)^{2l-2} \\ &\quad \leqslant \sum_{j=1}^{k} \frac{2\pi}{\alpha_j + \pi^2} \sum_{l=1}^{\infty} \frac{1}{2l+1} \left(\frac{2\pi\tau}{\alpha_j + \pi^2 + \tau^2}\right)^{2l-2} \\ &\quad \leqslant \omega_k \sum_{l=1}^{\infty} \frac{2\pi}{2l+1} \left(\frac{2\pi\tau}{\pi^2 + \tau^2}\right)^{2l-2}, \end{aligned}$$

one has (cf. (4.9))

$$0 \leqslant h_k(\tau) \leqslant g(\tau) + \omega_k h(\tau) \qquad (0 \leqslant \tau < \pi), \tag{4.11}$$

where the functions g and h are given by

$$\begin{aligned} g(\tau) &= 2 \sum_{l=1}^{\infty} \left(\frac{1}{\pi}\right)^{2l+1} \frac{\tau^{2l-2}}{2l+1}, \\ h(\tau) &= 4\pi \sum_{l=1}^{\infty} \frac{1}{2l+1} \left(\frac{2\pi\tau}{\pi^2+\tau^2}\right)^{2l-2}. \end{aligned} \tag{4.12}$$

Obviously, g and h are positive on $[0, \pi)$ and, moreover, $g(\tau) \to \infty$ and $h(\tau) \to \infty$ as $\tau \to \pi$. Let

$$\int_0^{\pi} \frac{1 - v_k(\tau)}{1 + v_k(\tau)} \frac{d\tau}{\sin(\tau/2)} = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_0^{\pi} \frac{e^{-\tau g_k(\tau)}}{1 + v_k(\tau)} \left(\frac{1 - e^{-\tau^3 h_k(\tau)}}{\tau}\right) \frac{\tau\, d\tau}{\sin(\tau/2)}, \\ I_2 &= \int_0^{\pi} \frac{1 - e^{-\tau g_k(\tau)}}{1 + v_k(\tau)} \frac{d\tau}{\sin(\tau/2)}. \end{aligned}$$

Using (4.10), the inequality $1 - e^{-t} \leqslant 2t(t+1)^{-1}$ $(t \geqslant 0)$, and the observation that

$$\frac{h_k(\tau)}{1 + \tau^3 h_k(\tau)} = \mathcal{O}(\omega_k) \qquad (k \to \infty),$$

uniformly on $[0, \pi)$, we may conclude that

$$I_1 = \mathcal{O}\left(\int_0^{\pi} \omega_k \tau^2 e^{-(2/\pi)\omega_k \tau} d\tau\right).$$

Hence

$$I_1 = \mathcal{O}(\omega_k^{-2}) \qquad (k \to \infty). \tag{4.13}$$

In a similar way one can prove that

$$I_2 = \int_0^{\pi} \frac{1 - e^{-\tau g_k(\tau)}}{1 + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} + \mathcal{O}(\omega_k^{-2}). \tag{4.14}$$

In view of (4.7) this leads to

$$\|\mathscr{S}_{2k+1}\| = \frac{1}{\pi}\int_0^\pi \frac{1-e^{-\tau g_k(\tau)}}{1+e^{-\tau g_k(\tau)}}\frac{d\tau}{\sin(\tau/2)}(1+\mathcal{O}(e^{-c\omega_k}))+\mathcal{O}(\omega_k^{-2}). \quad (4.15)$$

On account of (4.9) the function g_k may be written in the form

$$g_k(\tau)=\beta_k-\tau^2 r_k(\tau) \qquad (0\leqslant\tau<\pi), \quad (4.16)$$

where

$$\beta_k=\frac{2}{\pi}+4\pi\sum_{j=1}^{k}\frac{1}{\alpha_j+\pi^2}, \quad (4.17)$$

and

$$r_k(\tau)=4\pi\sum_{j=1}^{k}\frac{1}{(\alpha_j+\pi^2+\tau^2)(\alpha_j+\pi^2)}. \quad (4.18)$$

We observe that positive constants c_1 and c_2 exist such that $c_1\omega_k\leqslant\beta_k\leqslant c_2\omega_k$ ($k\in\mathbb{N}$). Therefore $\mathcal{O}(\omega_k^{-2})$ may be replaced by $\mathcal{O}(\beta_k^{-2})$, and vice versa.

From (4.18) it easily follows that

$$0<r_k(\tau)<\frac{\beta_k}{\pi^2+\tau^2} \qquad (k=1,2,...;\ 0\leqslant\tau<\pi). \quad (4.19)$$

Now, let

$$\int_0^\pi \frac{1-e^{-\tau g_k(\tau)}}{1+e^{-\tau g_k(\tau)}}\frac{d\tau}{\sin(\tau/2)}=J_1+J_2,$$

where

$$J_1=\int_0^\pi \frac{e^{-\beta_k\tau}(1-e^{\tau^3 r_k(\tau)})}{1+e^{-\tau g_k(\tau)}}\frac{d\tau}{\sin(\tau/2)},$$

$$J_2=\int_0^\pi \frac{1-e^{-\beta_k\tau}}{1+e^{-\tau g_k(\tau)}}\frac{d\tau}{\sin(\tau/2)}.$$

Using (4,19) together with the inequality $e^t-1\leqslant te^t$ ($t\geqslant 0$), we conclude that

$$\begin{aligned}J_1&=\mathcal{O}\left(\int_0^\pi \tau^2 e^{-\beta_k\tau}e^{\tau^3 r_k(\tau)}r_k(\tau)\,d\tau\right)\\&=\mathcal{O}\left(\int_0^\pi \tau^2 e^{-\beta_k\tau(1-\tau^2(\pi^2+\tau^2)^{-1})}r_k(\tau)\,d\tau\right)\\&=\mathcal{O}\left(\beta_k\int_0^\pi \tau^2 e^{-\beta_k\tau/2}d\tau\right)=\mathcal{O}(\beta_k^{-2})\end{aligned}$$

as $k \to \infty$. Similarly one has

$$J_2 = \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)} + \mathcal{O}(\beta_k^{-2}) \qquad (k \to \infty).$$

These relations for J_1 and J_2 yield (cf. (4.15))

$$\|\mathcal{S}_{2k+1}\| = \frac{1}{\pi} \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)} \left(1 + \mathcal{O}(e^{-c\omega_k})\right) + \mathcal{O}(\beta_k^{-2}). \qquad (4.20)$$

The integral in the right-hand side of (4.20) can be written as follows:

$$\int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)}$$

$$= \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{2}{\tau} \, d\tau + \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \left(\frac{1}{\sin(\tau/2)} - \frac{2}{\tau} \right) d\tau$$

$$= \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{2}{\tau} \, d\tau + \int_0^{\pi} \left(\frac{1}{\sin(\tau/2)} - \frac{2}{\tau} \right) d\tau + \mathcal{O}(\beta_k^{-2}).$$

The second integral can be evaluated quite easily; in fact

$$\int_0^{\pi} \left(\frac{1}{\sin(\tau/2)} - \frac{2}{\tau} \right) d\tau = 4 \ln 2 - 2 \ln \pi.$$

With respect to the first integral one has

$$\int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\tau} = \int_0^{\beta_k \pi} \frac{1 - e^{-\tau}}{1 + e^{-\tau}} \frac{d\tau}{\tau}$$

$$= \int_0^{\beta_k \pi} \tanh(\tau/2) \, d \ln \tau = \tanh(\beta_k \pi/2) \ln(\beta_k \pi)$$

$$- \frac{1}{2} \int_0^{\beta_k \pi} \frac{\ln \tau}{\cosh^2(\tau/2)} \, d\tau$$

$$= \ln(\beta_k \pi) - \frac{1}{2} \int_0^{\infty} \frac{\ln \tau}{\cosh^2(\tau/2)} \, d\tau + \mathcal{O}(e^{-d\omega_k}),$$

where d is a positive constant.

Using formula (4.371)(3) in Gradshteyn and Ryzhik [1, p. 580], we obtain

$$\int_0^{\infty} \frac{\ln \tau}{\cosh^2(\tau/2)} \, d\tau = 2(\ln \pi - \ln 2 - \gamma),$$

where γ denotes Euler's constant. We thus arrive at the following theorem.

THEOREM 4.1. *If* $\beta_k = 2/\pi + 4\pi \sum_{j=1}^{k} (\alpha_j + \pi^2)^{-1} \to \infty$ *as* $k \to \infty$, *then*

$$\|\mathscr{S}_{2k+1}\| = \frac{2}{\pi}(\ln \beta_k + 3 \ln 2 - \ln \pi + \gamma) + \mathcal{O}(\beta_k^{-2}) \qquad (k \to \infty). \tag{4.21}$$

Having dealt with $\omega_k \to \infty$ as $k \to \infty$, we now consider the case that the sequence (ω_k) is convergent.

II. $\lim_{k \to \infty} \omega_k < \infty$.

The convergence of the sequence (ω_k) implies that $\lim_{j \to \infty} \alpha_j = \infty$, so a positive integer k_0 exists such that $\alpha_j > 0$ for $j \geqslant k_0$. The polynomial q_{2k+1}, introduced at the beginning of this section, may therefore be written in the form

$$q_{2k+1}(z) = \gamma_k z^{2k_0 - 1} \left(1 + \frac{z^2}{\alpha_{k_0}}\right) \cdots \left(1 + \frac{z^2}{\alpha_k}\right) \qquad (k \geqslant k_0),$$

where

$$\gamma_k = \alpha_{k_0} \alpha_{k_0+1} \cdots \alpha_k.$$

Since $\sum_{j=k_0}^{\infty} \alpha_j^{-1}$ is finite, the product $\prod_{j=k_0}^{k} (1 + z^2 \alpha_j^{-1})$ converges uniformly in z on every bounded set of $\mathbb{C}$. As a consequence its limit function, which we denote by q, is a holomorphic function. Taking into account (4.1) we obtain

THEOREM 4.2. *If* $\sum_{j=1}^{\infty} (\alpha_j + 1)^{-1} < \infty$ *then*

$$\lim_{k \to \infty} \|\mathscr{S}_{2k+1}\| = \frac{1}{\pi} \int_0^{\pi} \frac{\sum_{m=-\infty}^{\infty} q^{-1}((2m+1)\pi - \tau)}{\sum_{m=-\infty}^{\infty} (-1)^m q^{-1}((2m+1)\pi - \tau)} \frac{d\tau}{\sin(\tau/2)}, \tag{4.22}$$

where

$$q(z) = \prod_{j=k_0}^{\infty} (1 + z^2 \alpha_j^{-1}).$$

Finally, we examine a few particular cases.

(a) *The polynomial case*: $\alpha_j = 0$ $(j = 1, 2, \ldots)$. Since $\beta_k = \pi^{-1}(2 + 4k) \to \infty$, Theorem 4.1 may be applied. A simple computation yields

$$\|\mathscr{S}_{2k+1}\| = \frac{2}{\pi}\left(\ln k + \ln \frac{32}{\pi^2}\right) + \mathcal{O}(k^{-1}) \qquad (k \to \infty),$$

which is in agreement with results obtained by Meinardus [2] and Richards [7].

(b) *The hyperbolic case*: $\alpha_j = j^2$ $(j = 1, 2, \ldots)$. Obviously $\sum_{j=1}^{\infty} (\alpha_j + 1)^{-1} < \infty$, and thus Theorem 4.2 may be applied. Using the well-known relation

$$\frac{\sinh(\pi z)}{\pi} = z \prod_{j=1}^{\infty} \left(1 + \frac{z^2}{j^2}\right),$$

we conclude from (4.22) that

$$\lim_{k \to \infty} \|\mathcal{S}_{2k+1}\| = \frac{1}{\pi} \int_0^{\pi} \frac{\sum_{m=-\infty}^{\infty} \sinh^{-1}(\pi((2m+1)\pi - \tau))}{\sum_{m=-\infty}^{\infty} (-1)^m (\pi((2m+1)\pi - \tau))} \frac{d\tau}{\sin(\tau/2)}.$$

A numerical computation of the integral yields

$$\lim_{k \to \infty} \|\mathcal{S}_{2k+1}\| \approx 2.1314.$$

References

1. I. S. Gradshteyn and I. M. Ryzhik, "Table of Integrals, Series, and Products," Academic Press, New York, 1980.
2. G. Meinardus, Ueber die Norm des Operators der Kardinalen Spline-Interpolation, *J. Approx. Theory* **16** (1976), 289–298.
3. G. Meinardus and G. Merz, Zur periodischen Spline-Interpolation, *in* "Spline-Funktionen, Vorträge und Aufsätze" (K. Böhmer *et al.*, Eds.), pp. 177–195, B.-I.-Wissenschaftsverlag, Mannheim, 1974.
4. C. A. Michelli, Cardinal *L*-splines, *in* "Studies in Spline Functions and Approximation Theory" (S. Karlin *et al.*, Eds.), pp. 203–250, Academic Press, New York, 1976.
5. H. G. ter Morsche, On the relations between finite differences and derivatives of cardinal spline functions, *in* "Spline Functions Karlsruhe 1975" (A. Dold and B. Ectmann, Eds.), pp. 210–219, Lecture Notes in Mathematics, Vol. 501, Springer-Verlag, Berlin, 1976.
6. H. G. ter Morsche, "Interpolational and Extremal Properties of *L*-Spline Functions," Thesis, Eindhoven University of Technology, Eindhoven, 1982.
7. F. Richards, The Lebesgue constants for cardinal spline interpolation, *J. Approx. Theory* **14** (1975), 83–92.
8. I. J. Schoenberg, On Michelli's theory of cardinal *L*-splines, *in* "Studies in Spline Functions and Approximation Theory" (S. Karlin *et al.*, Eds.), pp. 251–276, Academic Press, New York, 1976.